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ABSTRACT

This is one of a series of 20 booklets designed for participants in an in-service course for teachers of elementary mathematics. The course, developed by the University of Illinois Arithmetic Project, is designed to be conducted by local school personnel. In addition to these booklets, a course package includes films showing mathematics being taught to classes of children, extensive discussion notes, and detailed guides for correcting written lessons. This booklet contains exercises on rules moving two points and composition of number line rules, a summary of the problems in the film "Rules Moving Two Points," and the supplement. (MK)

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THE ARITHMETIC PROJECT COURSE FOR TEACHERS

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TOPICS: Rules Moving Two Points.
Composition of Number Line Rules.

FILM: Rules Moving Two Points, Grade 5

SUPPLEMENT: Examples of Questions Dealing With $\square \rightarrow \square \times \square$

NAME: .....

15

SE 031 115

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I. TWO POINTS TO TWO POINTS (Continued)

1. You know that $\square \longrightarrow 3 \times \square$ takes you from 5 to 15 and from 7 to 21. How could you change this rule so that $5 \longrightarrow 16$ and $7 \longrightarrow 22$?

2. Write a rule:

START		LAND
5	$\longrightarrow$	18
7	$\longrightarrow$	24

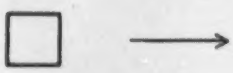
If you wrote $3 \times (\square + 1)$, you were correct. However, there is a simpler and, in the present context, more helpful way to write the rule: $\square \longrightarrow 3 \times \square + 3$.

3. Write a rule:

START		LAND
5	$\longrightarrow$	21
7	$\longrightarrow$	27

4. Write a rule:

START		LAND
5	$\longrightarrow$	0
7	$\longrightarrow$	6



5. Write a rule:

START		LAND
5	$\longrightarrow$	$-2\frac{1}{4}$
7	$\longrightarrow$	$3\frac{3}{4}$

	Starting Points	Landing Points	1st Starting pt. minus 2nd Starting pt.	1st Landing pt. minus 2nd Landing pt.	Rule
1.	6	18	6	18	□ →
	0	0			
2.	6	17			
	0	-1			
3.	6	_____		18	
	0	982			
4.	8	32			
	1	4			
5.	8	38			
	1	10			
6.	8	_____		28	
	1	101			
7.	23	_____			□ → □ × 5
	16	_____			
8.	23	226			
	16	191			
9.	1	8			
	2	13			
10.	12	25			
	6	13			
11.	12	23			
	6	11			
12.	12	24			
	6	0			
13.	12	120			
	6	0			

	Starting Points	Landing Points	1st Starting pt. minus 2nd Starting pt.	1st Landing pt. minus 2nd Landing pt.	Rule
1.	7	→ 16			
	$4\frac{1}{2}$	→ 1			
2.	13	→ $43\frac{1}{2}$			
	6	→ $1\frac{1}{2}$			
3.	300	→ _____			$\square \rightarrow \frac{1}{3} \times \square$
	150	→ _____			
4.	_____	→ 100			$\square \rightarrow \square \div 3^{\dagger}$
	_____	→ 50			
5.	_____	→ 50			$\square \rightarrow \frac{1}{6} \times \square$
	_____	→ 25			
6.	300	→ 57			
	150	→ 32			
7.	300	→ $62\frac{3}{4}$			
	150	→ $37\frac{3}{4}$			
8.	300	→ _____			$\square \rightarrow \frac{1}{6} \times \square - 54$
	150	→ -29			
9.	$29\frac{1}{2}$	→ Leave Blank		2	
	$17\frac{1}{2}$	→			
10.	30	→ 45			
	20	→ 30			
11.	30	→ 116			
	20	→ 101			
☆ 12.	30	→ $27\frac{1}{2}$			
	20	→ 20			

$^{\dagger}\square \rightarrow \square \div 3$ is the same rule as $\square \rightarrow \frac{1}{3} \times \square$.

☆1. In a student's words, what is a general procedure for finding a rule that takes you from two given starting points on a number line to two given landing points?

☆2. Write a rule:

START		LAND
11	⟶	10
17	⟶	30

II. COMPOSITION

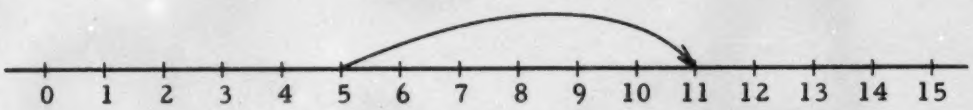
Occasionally an incident like the following will occur in a classroom:

Teacher: Start at 5 and make one jump with the rule $\square \longrightarrow 3 \times \square - 4$.

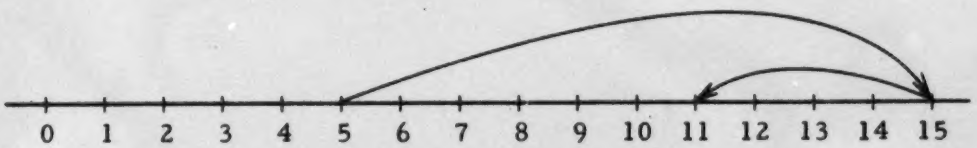
Student (thinking aloud):

Well, 3 times 5 is 15, so first you land on 15. Then 15 minus 4 is 11. So if you start on 5 you land on 11.

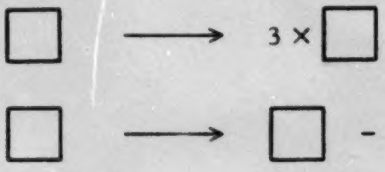
The answer 11 is right, of course, but this jump never involved landing on 15. If one were to draw a picture of this jump, one would draw



and not

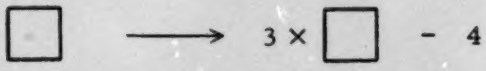


On the other hand, the child who draws the two arrows has sensed a very important thing. If one were to use each of the rules

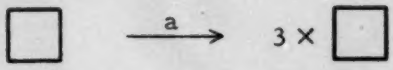


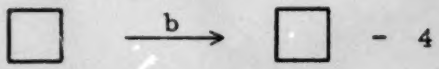
and

one right after the other, the result is the same as using just once the rule

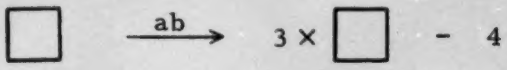


If we label the two simpler rules a and b like this:

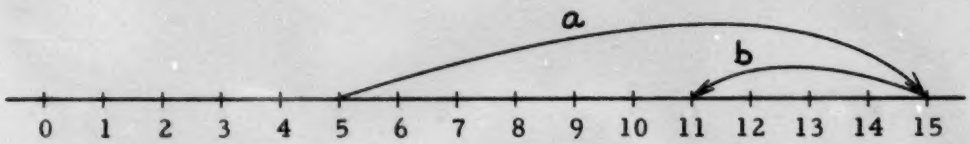


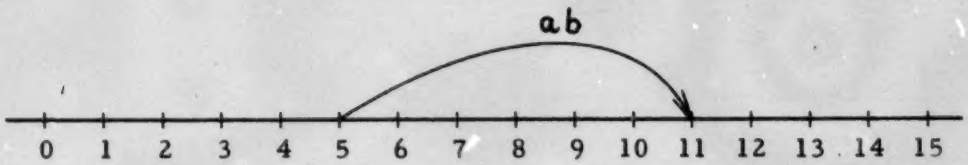


then it makes sense to call the more complicated rule ab :



The number lines below show the same jumps we had before.





1. If rules c and d are

$\square \xrightarrow{c} 7 \times \square$

$\square \xrightarrow{d} \square + 8\frac{1}{2}$

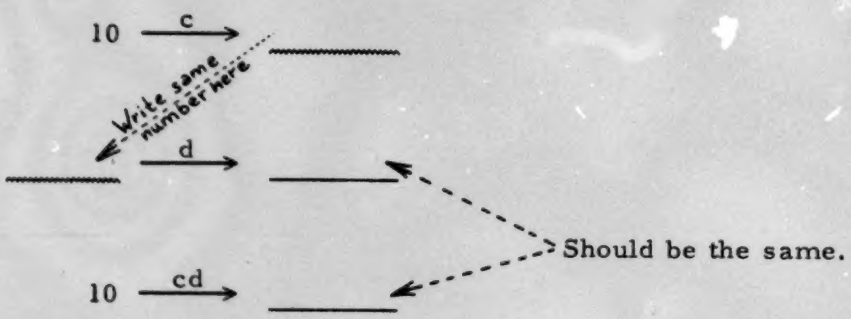
write a single rule (called cd) which has the same effect as using first rule c and then rule d .

$\square \xrightarrow{cd}$ _____

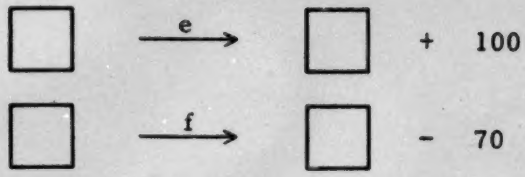
Start at 2 and make one jump with rule c . Where do you land? _____
Starting at this new number make one jump using rule d . Where do you land? _____

Now make one jump with your rule cd , starting back at 2. Where do you land? _____ (Your last two answers should be the same.)

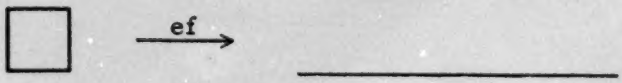
Another check of your rule cd . Use the indicated rules to fill in the blanks with numbers.



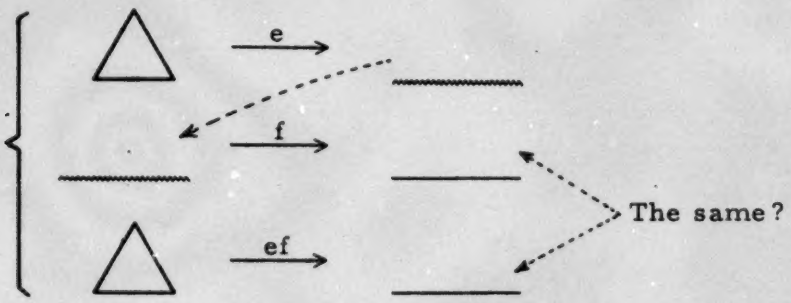
2. Rules e and f are



Write rule ef :

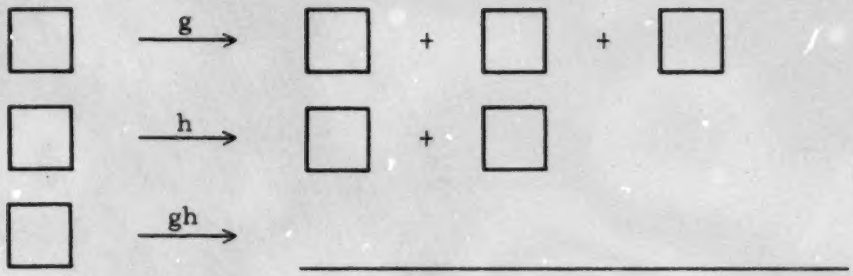


As a check, try a number in wedge:

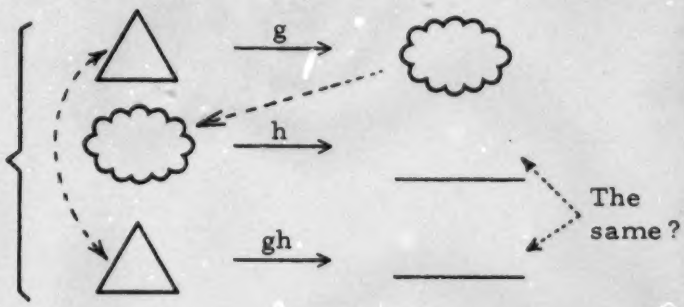


(Often one test number and insight will convince you beyond doubt. Otherwise test with more starting numbers. You should be sure your proposed rule works for <u>all</u> starting numbers.)

3. Write rule gh :



Pick one number for both wedges. Compute the number for cloud using rule g . Put the same number in both clouds. Find the indicated landing points.



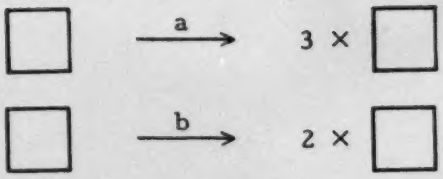
If you got □ ⟶ 5 × □ , for rule gh , you have the most likely wrong answer, and this paragraph is for you. Try starting at 5, for example. You can now compute

$$5 \xrightarrow{g} 15$$

and $$15 \xrightarrow{h} 30$$

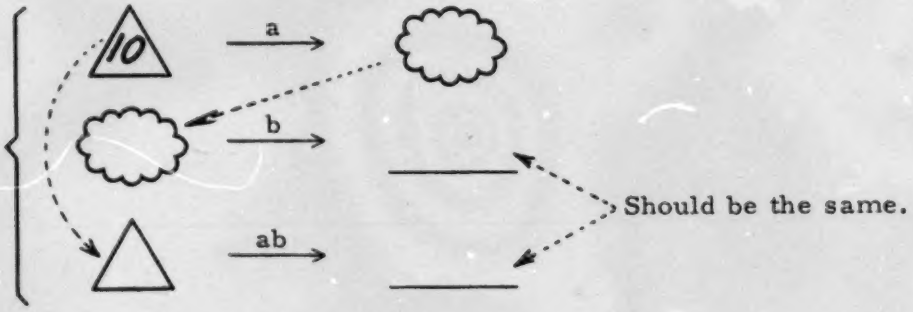
If your rule gh were correct, it would give you 30 when you start on 5. But it doesn't; it takes you to 25 instead. Can you fix it? Rule gh is the same as first tripling a number and then doubling the result. This is the same as multiplying it by _____ Rule gh is □ ⟶ _____ × □ . Check your answer on page 10.

4. New rules:

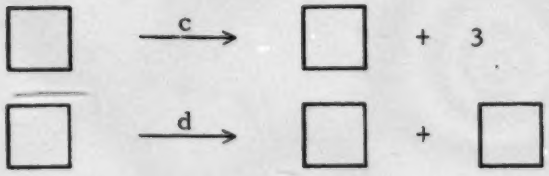


Write rule ab :

Check by starting at 10.

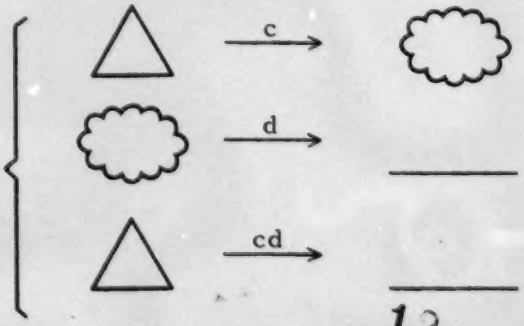


5. New rules:

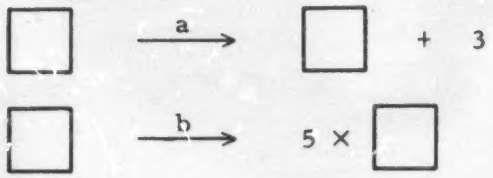


Write rule cd :

(Hint: The rule cd is <u>not</u> □ ⟶ 2 × □ + 3, nor is it □ ⟶ 3 × □ + 3.)
Check your result with one or more numbers as before.



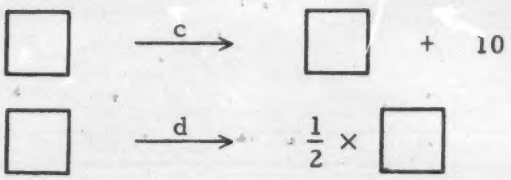
6. Rules:



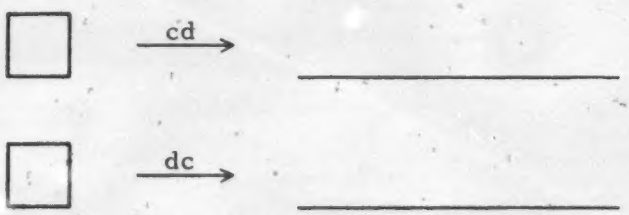
Write rule ab :

Check your rule with one or more numbers.

7. Rules:



Write rules cd and dc :



(Note that rule dc is very different from rule cd . If in doubt check both rules.)

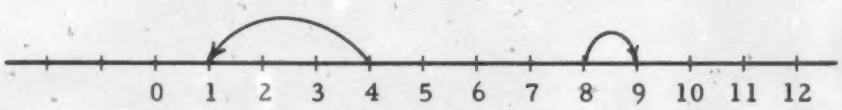
Answer for page 8: Rule gh is □ ⟶ 6 × □.

Epilogue

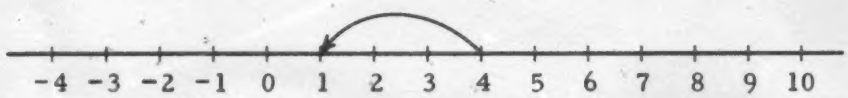
You may have wondered whether there is always only one rule which takes two given points to two given points. To answer this question consider the problem of finding a rule with these jumps:

START		LAND
4	⟶	1
8	⟶	9

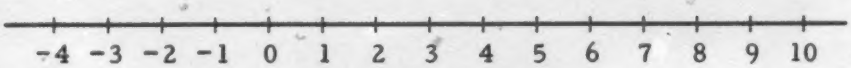
One way to picture these jumps is



However, there is another way of showing starting and landing numbers. If we want to show the jump from 4 to 1, we place a dot 1 unit above 4 on the number line. So instead of drawing



we draw a picture like this:



To make it easier to put the dot at the correct height we make a vertical number line through zero and draw lighter lines vertically and horizontally as guides:

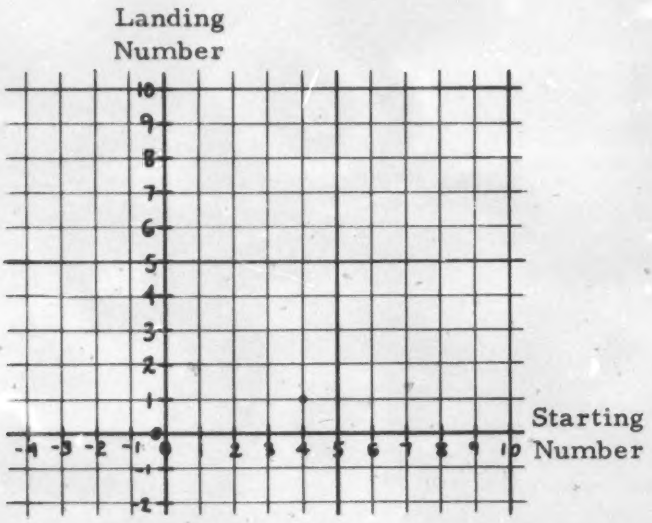


Fig. 1

The single dot in Figure 1 stands for the jump from 4 to 1.

The next picture shows the points (4, 1) and (8, 9) marked. They show the two jumps that we want our rule to have.

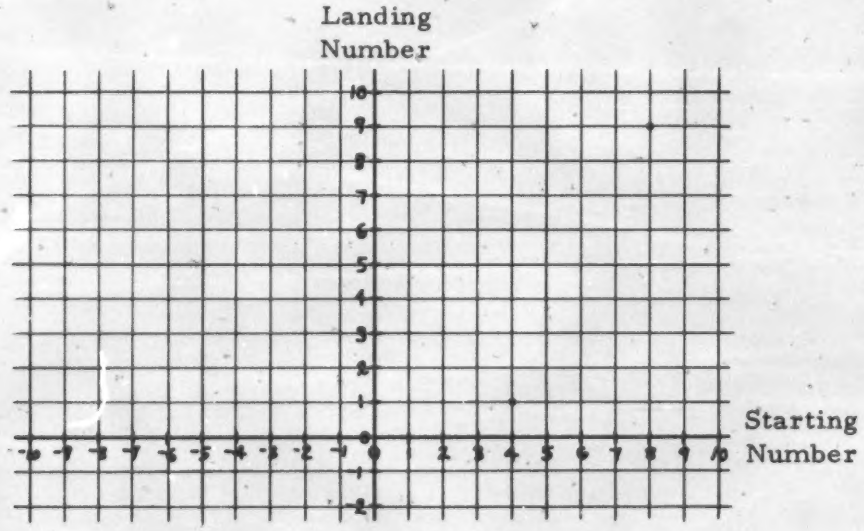


Fig. 2

Geometry tells us that through two given points one and only one straight line can be drawn. Draw the one line that joins the two dots in Figure 2. That line depicts many jumps with the rule $\square \longrightarrow 2 \times \square - 7$. The fact that there is only one line through the two points tells us that there is only one linear* rule that has jumps from 4 to 1 and from 8 to 9.

The number plane may help you to imagine that there may be other more complicated rules which have the two required jumps. The next picture shows jumps with the rule $\square \longrightarrow \square \times \square - 10 \times \square + 25$. (This is not a linear rule because of the $\square \times \square$.) If you test this rule you will find that $4 \longrightarrow 1$ and $8 \longrightarrow 9$.

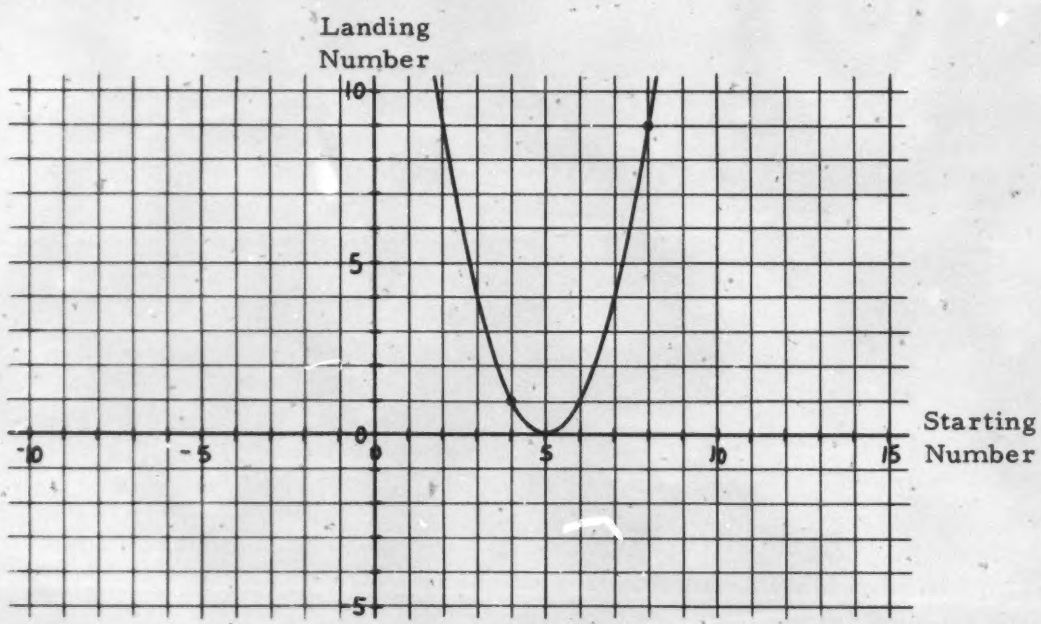


Fig. 3

Rules in which $\square$ is multiplied by $\square$ always have curves something like the one above. When a rule involves $\square \times \square \times \square$, the curve gets more complicated. You need not be concerned with such rules now.

---

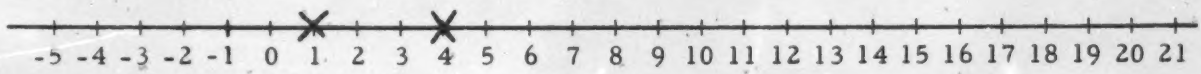
*A linear rule is one that can be written in the form $\square \longrightarrow a \times \square + b$, where a and b are numbers. As you may have guessed, the graph of a linear rule is a straight line.

Summary of Problems in the Film

"Rules Moving Two Points"

5th Grade, James Russell Lowell School, Watertown, Massachusetts
Teacher: David A. Page

Note: This summary includes the relevant section of the preceding day's class, and all the problems given during the filmed class. Sections that appear here in dotted boxes are those which are not included in the film.



Now we're going to take both of these x's, use one number line rule, jump them to a new location, and find out how far apart they are then.

How far between the x's? (3 units)

Here's a rule: $\square \longrightarrow 4 \times \square$

What happens to 1 using this rule? (1 goes to 4, so the rule is discarded as confusing)

Change rule to $\square \xrightarrow{a} 5 \times \square$

What happens to 1 with this rule? (Lands at 5)

What happens to 4 ?

How far between 1 and 4 ? (3 units)

How far between 5 and 20 ?

New rule: $\square \xrightarrow{b} 2 \times \square$

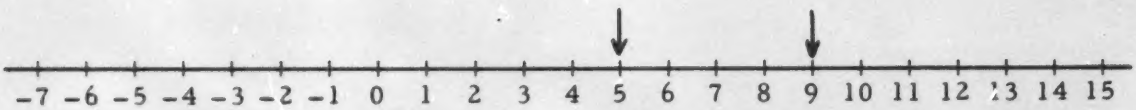
What happens if we put 1 in rule b ? (Go to 2)

What happens to 4 ?

How far from 2 to 8 ? (6 units)

Using the rule $\square \xrightarrow{c} 3 \times \square$, how far apart do you predict the landing points will be? (9 units)

(Next day.)



New rule: $\square \xrightarrow{a} 2 \times \square$

Put 5 through that rule, where does it go? (10)

Put 9 through the rule, where does it go?

How far between 5 and 9 ?

How far between 10 and 18 ?

Give a rule b that will give landing points 12 units apart when we start with 5 and 9 . (Answer: $\square \xrightarrow{b} 3 \times \square$)

Put 5 in this rule, where does it go?

Put 9 in this rule.

How far is it between 15 and 27 ?

Now get a rule using 5 and 9 so that the landing points will be 2 units apart. (Two answers: $\square \xrightarrow{c} \frac{\square}{2}$; $\square \xrightarrow{d} \frac{1}{2} \times \square$)

Put 5 in rule c , land?

Put 9 in rule c , what do you get?

How far between $2\frac{1}{2}$ and $4\frac{1}{2}$?

Using rule d , where does 5 go? ($2\frac{1}{2}$)

Get a rule that keeps the two points four units apart and makes them both negative. ("Any number minusing above 10 ," e.g., $\square \longrightarrow \square - 11$)

Using the rule $\square \longrightarrow \square - 11$, starting at 5 what happens?
(Land at -6)

Where do you go starting at 9 ? (-2)

New rule: □ ⟶ 3 × □ − 10

Predict what this rule is going to do.

(Some predictions: "Every two points that you put in end up two units apart." "That's not true because there is a standstill point and that's no units apart.")

What is the standstill point? (5)

Using 5 and 8 for the two points, where does 5 go? (5)

Where does 8 go? (14)

How far from 5 to 14 ?

How far from 5 to 8 ?

If we picked two other points, what do you think would happen?

Try 15 and 20 . How far apart are 15 and 20 ? (5 units)

Put 15 in the rule, where does it go? (35)

Put 20 in the rule, where does it go? (50)

How far apart are 35 and 50 ? (15 units)

What is happening?

"Every answer will come out an odd number. From 15 to 20 is five and that's an odd number; and 5 to 14 is nine units and that's an odd number."

"You mean if we start out at an odd number for the distance, we end up with an odd number for the distance? Is he right, Janet?"

"Yes, and if you put in an even number, it will come out to be an even number."

"Do you believe that, Andrea?"

"I have my own solution: 5 to 8 is three units apart and when you did the whole rule, the number of units between the 2 points was 3 times the three [units], and that was nine units, and that's what happened. With the 15 and 20 , 15 went to 45 subtract 10 is 35 . 20 went to 50 and then— that 15 and 20 were five units apart and five times 3 is fifteen [units]. That's how you get how many units they will be apart."

Change the rule a little bit so that if we start out with 5 and 8, we end up with points 9 units apart, but one of the points is 0.

(Wrong answer of $\square \longrightarrow 2 \times \square - 10$;
Correct answer: $\square \longrightarrow 3 \times \square - 15$)

With the rule $\square \longrightarrow 3 \times \square - 15$, 5 goes to 0. Make up another rule so that 8 is the one that goes to 0.

(Wrong answers: $\square \longrightarrow 3 \times \square + 15$, $\square \longrightarrow \frac{3 \times \square}{3} - 8$

Correct answer: $\square \longrightarrow 3 \times \square - 24$)

Using the rule $\square \longrightarrow 3 \times \square - 24$, where does 5 go? (−9)

Where does 8 go?

How far apart are 0 and −9 ?

Now the starting points are 0 and 1. Write down a rule that gets landing points that are 6 units apart and one of them is 10.

(Answer: $\square \longrightarrow 6 \times \square + 4$)

Using this rule, what happens to 0 ? (Lands at 4)

What happens to 1 ?

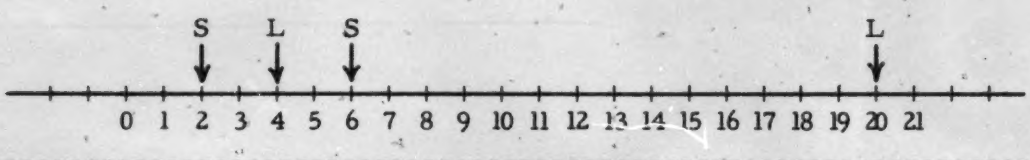
How far apart are 4 and 10 ?

What is the other rule that does it? ($\square \longrightarrow 6 \times \square + 10$)

Put 1 in this rule, what happens? (Lands at 16)

How far apart are 10 and 16 ?

Now the two starting points are 2 and 6, and the two landing points are 4 and 20. Make up a rule that will get you from the two starting points to the two landing points.



(Ans: $\square \longrightarrow 4 \times \square - 4$)

Using $\square \longrightarrow 4 \times \square - 4$, let's try 2.

What about 6 ?

Now find the rule that will take you from 2 to 20 and 6 to 4.

Supplement

Examples of Questions Dealing With $\square \longrightarrow \square \times \square$

In a recent written lesson you were asked to write a sequence of questions introducing the rule $\square \longrightarrow \square \times \square$. We thought you might be interested in seeing some ways this question has been answered by others.

The first four of these examples were written by participants in an institute such as yours. The remaining four are the individual responses of four Arithmetic Project staff members. The variety of possible styles in approaching such a question is particularly noteworthy.

Institute participants' work has been retyped for clarity, but in most cases their notation has not been altered. You may find that some ways of writing questions and answers seem clearer than others. Where errors occurred the corrector's comments are included in a footnote.

Assume you teach a fifth or sixth grade, and you have worked with problems involving the multiplication of negatives (i.e., your class has done such things as $-4 \times -4 = 16$).

Now consider the rule

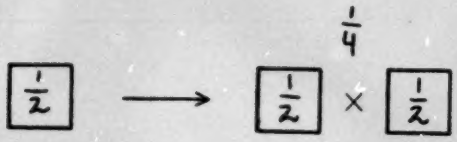
Write 10 or more specific questions or problems (and give the answers) that you might use to enable your students to make some generalizations about this rule. You want them to find out such things as these: From what starting points do jumps with this rule go to the right? To the left? Are there regions you can't land on at all? What about standstill points?

SAMPLE A

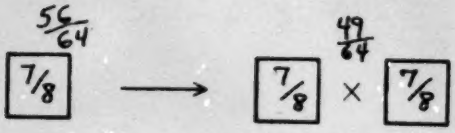
Using above rule:

3 →	9 3 × 3	goes to the right	1. Any negative number or any positive number greater than 1 when multiplied by itself—jumping rule goes to right.
8 →	64 8 × 8	" " " "	
-8 →	+64 -8 × -8	" " " "	
-14 →	+196 -14 × -14	" " " "	
-7 →	+49 -7 × -7	" " " "	
-1 →	+1 -1 × -1	" " " "	2. −1 goes to the right. +1 standstill. Also 0.
+1 →	+1 × +1	Standstill (Also try others.)	

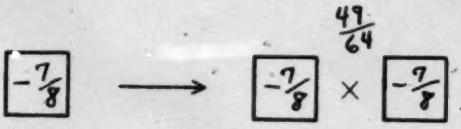
SAMPLE A, continued



goes to the left



" " " "



goes to the right

3. Positive fractions go to the left, negative fractions go to the right.

4. Unable to land in negative regions because $- \times - = +$

SAMPLE B

1. $5 \longrightarrow 5 \times 5 = 25$
2. $-1 \longrightarrow -1 \times -1 = 1$
3. $-4 \longrightarrow -4 \times -4 = 16$
4. $-5 \longrightarrow -5 \times -5 = 25$
5. $-20 \longrightarrow -20 \times -20 = 400$

6. Can we ever land on -25 with this rule? Why or why not?
   (No. Because a negative times a negative gives a positive.)

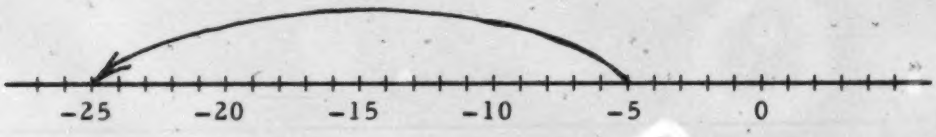
7. Using this rule will we ever go to the left on the number line? (No)*

8. What number is a standstill point?
   ($0 \longrightarrow 0 \times 0 = 0$) Can there be any others? (No.)†

9. Can you change the rule so it would be possible to land on -25 ?

$-5 \longrightarrow -5 \times 5 = -25$

10. In what direction do we move using this new rule and starting at -5 ?
    (Left. Show this on the number line.)



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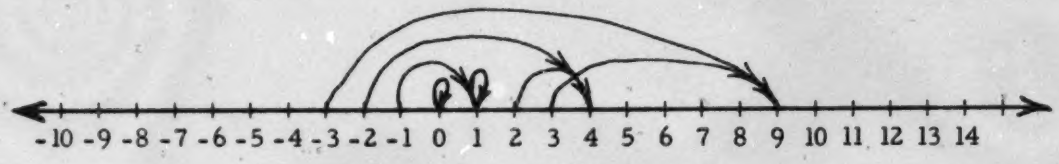
* The answer to this question is, of course, incorrect. The corrector wrote: "In question 7, try starting at $\frac{1}{2}$, $\frac{1}{8}$, $\frac{9}{10}$, $\frac{7}{8}$, etc."

† The corrector wrote: "Try 1."

SAMPLE C

Consider the rule $\square \longrightarrow \square \times \square$. Use the numbers below and make one jump with each. Mark each starting place and each landing place on the number line.



Start	Land	Distance
2	4	2
-2	4	6
3	9	6
-3	9	12
0	0	0
+1	+1	1*
-1	+1	2

* The corrector wrote: "This should be 0."

SAMPLE C, continued

Questions:

A. What numbers can be used in the rule to make jumps to the right? (All numbers.)*

B. Who can name a starting place that would allow us to move to the left? (There are none. However, try every number a child might give.)*

C. From examining the above jump problems, who can name a standstill point? (1)

D. Are there any other standstill points? (Yes.) Name them. (0)

* The answers to A and B are incorrect. The corrector wrote: "Try $\frac{1}{2}$ and other fractions."

SAMPLE D

1. $\boxed{2} \longrightarrow \boxed{2} \times \boxed{2} = 4$ We move to the right.

$\boxed{-2} \longrightarrow \boxed{-2} \times \boxed{-2} = 4$ " "

$\boxed{3} \longrightarrow \boxed{3} \times \boxed{3} = 9$ " "

$\boxed{-3} \longrightarrow \boxed{-3} \times \boxed{-3} = 9$ " "

$\boxed{-\frac{1}{2}} \longrightarrow \boxed{-\frac{1}{2}} \times \boxed{-\frac{1}{2}} = \frac{1}{4}$ " "

$\boxed{\frac{1}{2}} \longrightarrow \boxed{} \times \boxed{} =$ We move to the left.

It would seem that all starting places make us move to the right except positive fractions.

2. Where will you land?

Start	Land	
-3	9	Our numbers get smaller
-2	4	
-1	1	
1	1	larger
2	4	
3	9	

Where does the pattern change? (Starting points from 1 to -1.)

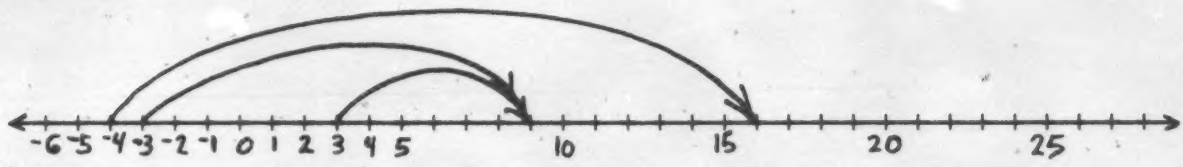
Can you find a starting place where we will have to land right back on the same number? (0) Is there another? (1) Why not -1? (We would get a positive answer.)

3. Where will I have to start to land on

4 ?	(2 or -2)
5 ?	($\sqrt{5}$ or $-\sqrt{5}$)
1 ?	(1 or -1)
0 ?	(0)
$\frac{1}{4}$?	($\frac{1}{2}$ or $-\frac{1}{2}$)
$-\frac{1}{4}$?	(impossible)

(Neither a negative number squared nor a positive number squared can possibly give a negative answer.)

SAMPLE E



Start at 3. Where do you land? (9)

Start at −4. Where do you land? (16)

3 landed at 9. Find another starting place so that you land at 9. (−3)

Where can we start to land on 121? (11)

Another place to start to land on 121? (−11)

Where can we start to land on −4? (You cannot land on −4)

Why? (Because you can only land on positive numbers and zero)

Start at $2\frac{1}{2}$. Where do you land? $(6\frac{1}{4})$

If you start on $-4\frac{1}{4}$, will the jump be longer or shorter than 20 spaces? (Longer) How do you know? (The jump from −3 goes to 9 and the jump from −4 goes to 16, so the farther you go down the line the longer the jump. Since the jump from −4 is 20 spaces the jump from $-4\frac{1}{2}$ will be longer.)

Where will you land if you start at $-4\frac{1}{4}$? $(18\frac{1}{16})$

All the jumps so far have gone to the right. Find a place to start so the jump goes to the left. (Any fraction between 0 and 1)

Where are all the starting places so that we go to the left? (Any fraction between 0 and 1)

Start at $\frac{1}{2}$ and think of making lots and lots of jumps. How many jumps will you have to make so that you land on 0? (You never will land on 0.)

SAMPLE E, continued

Where can you start to land on 0? (0)

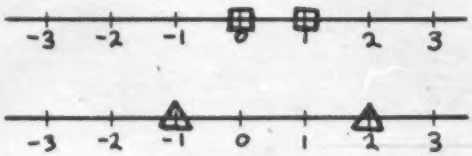
Find another standstill point. (1)

Is there another? (No)

Why? (Any negative number will land in the positives. Any number bigger than 1 times itself will be bigger than itself and any fraction times itself will be smaller.)

So the only standstill points are 0 and 1. Can you change the rule by adding or subtracting something so the standstill points are −1 and 2 ?

$\square \longrightarrow \square \times \square$



$\square \longrightarrow ?$

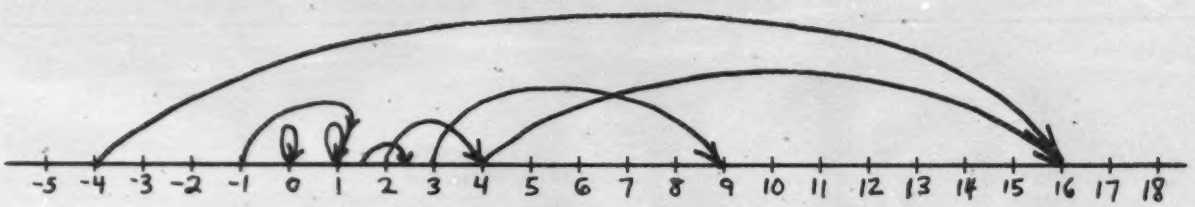
$\square \longrightarrow \square \times \square - 2$

If the standstill points are −2 and 3, what is a rule? $\square \longrightarrow \square \times \square - 6$

−4 and 5 ? $\square \longrightarrow \square \times \square - 20$

−99 and 100 ? $\square \longrightarrow \square \times \square - 9{,}900$

SAMPLE F



Start at 3. Land: ____ (9)

Start at 4. Land: ____ (16)

Where else can you start to land on 16? (−4)

Where can you start to land on 64? (8, −8) On 10,000? (100, −100)

T: Where can you start to get a smaller jump than any so far?

S: 2. $2 \longrightarrow 4$. That's only 2 spaces.

S: −1. $-1 \longrightarrow 1$. And that's also 2 spaces.

S: $1\frac{1}{2}$. $1\frac{1}{2} \longrightarrow 2\frac{1}{4}$. That's less than 2 spaces.

T: Can you find a standstill point?

S: 0.

S: Also 1.

T: So far, all jumps have gone to the right. Can you find a starting place so jumps go to the left?

S: No. Since this rule multiplies numbers by themselves, you'll either stand still or get bigger. So you have to go <u>up</u> the line.

S: What happens if you start at $\frac{1}{2}$?

S: You'll get to $\frac{1}{4}$, and that goes backwards.

T: Where else can you start to "go backwards"?

S: Try $\frac{1}{3}$. ($\frac{1}{3} \longrightarrow \frac{1}{9}$)

S: Try $1\frac{1}{3}$. ($1\frac{1}{3} \longrightarrow \frac{16}{9}$, or $1\frac{7}{9}$, so that goes to the right.)

S: Try $\frac{1}{4}$. ($\frac{1}{4} \longrightarrow \frac{1}{16}$, so that's okay.)

S: Try $-\frac{1}{2}$.

S: No, that won't work because $-\frac{1}{2} \times -\frac{1}{2}$ is going to be positive, and the jump will go right. In fact, we don't have to try any negative numbers because they'll all go right.

SAMPLE F, continued

T: Where are all the places you can start to get a jump that goes to the left?

S: It looks as if any number between the standstill points will do it.

T: Why?

S: When you multiply a fraction between 0 and 1 by itself, you'll get a smaller fraction.

T: Right. What do you think will happen if we change the rule to:

$$\square \xrightarrow{b} \square \times \square + 1$$

SAMPLE G

The first time I introduced my class (6th grade) to this rule I used rule a as $\square \xrightarrow{a} \square \times \square$ and rule b as $\square \xrightarrow{b} \square$ and we determined which rule won for different starting numbers.

Start at 10. $10 \xrightarrow{a} 100$, $10 \xrightarrow{b} 10$ — a wins

Start at −10. $-10 \xrightarrow{a} 100$, $-10 \xrightarrow{b} -10$ — a wins

Where do they tie? 0 as first answer, then someone said 0 was the only tie.

Start at 2. $2 \xrightarrow{a} 4$, $2 \xrightarrow{b} 2$ — a wins

Start at −1. $-1 \xrightarrow{a} 1$, $-1 \xrightarrow{b} -1$ — a wins

At this point someone said a would always win except at 0. Someone else said start at 1. $1 \xrightarrow{a} 1$, $1 \xrightarrow{b} 1$ — tie at 1

Can anyone find a place where b will win?

Suggestion: $-\frac{1}{3}$ $-\frac{1}{3} \xrightarrow{a} \frac{1}{9}$, $-\frac{1}{3} \xrightarrow{b} -\frac{1}{3}$ — a wins

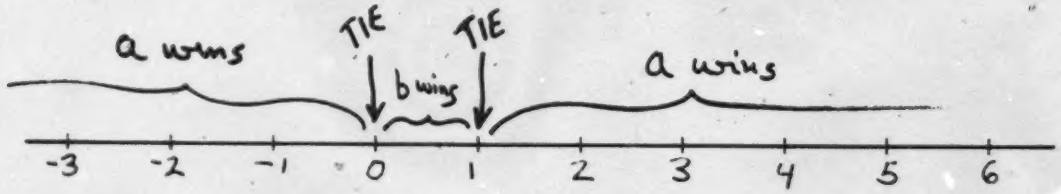
Then $\frac{1}{6}$ $\frac{1}{6} \xrightarrow{a} \frac{1}{36}$, $\frac{1}{6} \xrightarrow{b} \frac{1}{6}$ — b wins

Suggestion: $2\frac{1}{3} = \frac{7}{3}$ $\frac{7}{3} \xrightarrow{a} \frac{49}{9}$ or $5\frac{4}{9}$, $\frac{7}{3} \xrightarrow{b} \frac{7}{3}$ or $2\frac{1}{3}$ — a wins

Then $\frac{2}{3}$ b wins

SAMPLE G, continued

Generalization: b wins for every fraction between 0 and 1.



Next we compared rules

$\square \xrightarrow{a} \square \times \square$

$\square \xrightarrow{b} 2 \times \square$

and, on another day,

$\square \xrightarrow{a} \square \times \square$

$\square \xrightarrow{b} \square \times \square \times \square$

SAMPLE H

(Note: All jumps would be drawn on number line.)

Start at 5. Make one jump. Land? [25]

Start at 7. Make one jump. Land? [49]

Start at −3. Make one jump. Land? [9]

(Digression for -3×-3 if necessary.)

Start somewhere, make a jump and land on 100. Where did you start? [10] Could you have started somewhere else and landed on 100? [Yes] If so, where? [−10]

Start somewhere, make one jump and land on −16. Where did you start? [Impossible]

(Digression for checking 4 and −4 if necessary.)

Which starting number will give a longer jump, 4,382 or −4,382? [−4,382] Why? [$(4,382)^2 - (-4,382) > (4,382)^2 - (4,382)$]

Notice that all jumps so far have gone to the right. Where can we start to go to the left? [Between 0 and 1] If no ideas: Suppose someone said that there was a negative starting number that would give a jump to the left. How would you prove that he was wrong? [If $\square < 0$ then $\square^2 > 0$ and so $\square^2 > \square$.] Suppose someone said that there was a number around 100 that would give a jump to the left. How would you prove that he was wrong? [Starting numbers around 100 would give landing numbers around 100^2 or 10,000. This is going to the right.] If still no ideas on jumps to the left: Is there a place you can start so that the jump is neither to the right nor to the left? [0 and 1] Where could you start to get a very short jump? [Close to 0 and close to 1] Try some—e.g., $\frac{1}{10}$, $-\frac{1}{10}$, $\frac{9}{10}$, $1\frac{1}{10}$ [getting $\frac{1}{100}$, $\frac{1}{100}$, $\frac{81}{100}$, $1\frac{21}{100}$ respectively]. Which ones go to the left? [The jumps from $\frac{1}{10}$ and $\frac{9}{10}$.] How long are these jumps? [$\frac{9}{100}$, $\frac{11}{100}$, $\frac{9}{100}$ and $\frac{11}{100}$ respectively].

SAMPLE H, continued

Of all the jumps that go to the right, which is longest? [There is none.]

" " " " " " " " right " " shortest? [There is none.]

" " " " " " " " left " " longest? [Start: $\frac{1}{2}$]

" " " " " " " " left " " shortest? [There is none.]

What makes you think starting at $\frac{1}{2}$ gives the longest jump to the left?

Various answers could be given. You may find a proof difficult, but the answer might be justified with varying degrees of rigor as follows:

1) $\frac{1}{2}$ is right in the middle between 0 and 1 .

2) $\frac{1}{10}$ and $\frac{9}{10}$ gave jumps of same length. So will $\frac{1}{100}$ and $\frac{99}{100}$, $\frac{1}{3}$ and $\frac{2}{3}$, $\frac{4}{10}$ and $\frac{6}{10}$, etc. The closer you get to $\frac{1}{2}$ the bigger the jumps get. $\frac{1}{2}$ will give biggest jump.

3) The jumps from $\frac{49}{100}$ and $\frac{51}{100}$ are shorter than the one from $\frac{1}{2}$.

4) If we put any number which is close to zero (either pos. or neg.) in Δ, then the rule takes $(\frac{1}{2} + \Delta)$ to $(\frac{1}{2} + \Delta)^2$ or $\frac{1}{4} + \Delta + \Delta^2$. The length of this jump is $\frac{1}{2} + \Delta - (\frac{1}{4} + \Delta + \Delta^2)$ or $\frac{1}{4} - \Delta^2$. Since $\Delta^2 > 0$, $\frac{1}{4} - \Delta^2 < \frac{1}{4}$. But the jump from $\frac{1}{2}$ is of length $\frac{1}{4}$. Therefore, the jump from $\frac{1}{2}$ is longer than any other starting between 0 and 1 since we can put any number in Δ as long as $-\frac{1}{2} < \Delta < \frac{1}{2}$.

SAMPLE H. continued

5) Here is exactly the same argument as in 4) above, in different (and more) words.

Consider the jump starting at $\frac{1}{2}$. We land on $\frac{1}{4}$, so the length of the jump is $\frac{1}{4}$. We want to know how the length of the jump changes if we start a little bit above or below $\frac{1}{2}$.* So we think of starting at $\frac{1}{2} + \Delta$, where we can put positive or negative numbers (between $-\frac{1}{2}$ and $\frac{1}{2}$) in the wedge. For the rest of this discussion we will <u>not</u> put 0 in Δ because we already know what happens when we start at $\frac{1}{2} + 0$, or $\frac{1}{2}$.

We now have a jump starting at $\frac{1}{2} + \Delta$. This lands at $(\frac{1}{2} + \Delta)^2$, which, when you multiply it out, is $\frac{1}{4} + \Delta + \Delta^2$. The length of this jump is the difference between starting point and landing point. Thus the length is

$$\frac{1}{2} + \Delta - (\frac{1}{4} + \Delta + \Delta^2)$$

or $$\frac{1}{2} + \Delta - \frac{1}{4} - \Delta - \Delta^2$$

or $$\frac{1}{4} - \Delta^2$$

We know that $\Delta^2 > 0$ no matter what we put in Δ. (Remember we are not putting 0 in Δ.) Also we know that Δ^2 must be less than $\frac{1}{4}$, because the only numbers we are putting in Δ are those such that $-\frac{1}{2} < \Delta < \frac{1}{2}$.

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*Starting anywhere else where the jumps go to the left, i.e., starting anywhere between 0 and 1 (other than at $\frac{1}{2}$ itself, of course).

SAMPLE H, continued

So $\triangle^2$ is above 0 but is less than $\frac{1}{4}$. This makes the length, $\frac{1}{4} - \triangle^2$, always come out less than $\frac{1}{4}$. So all the jumps starting above or below $\frac{1}{2}$ in this region between 0 and 1 have lengths of less than $\frac{1}{4}$. Thus it must be that the jump starting <u>at</u> $\frac{1}{2}$, which we have already found to have a length of $\frac{1}{4}$, is longer than any of the others. And since this is the only region where there are jumps to the left, the jump from $\frac{1}{2}$ is the longest jump to the left.